Proposed Summer Research in Support Theory

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1 Summary of Proposed Research

Preconditioning is an approach to solving the matrix equation $Ax = b$ numerically. Recent work has shown a useful relationship between a factorization of a matrix, its orthogonal projector, and a factorization of its generalized inverse. Professor Guattery and I would investigate whether this factorization can be used to generate sparse and useful approximate inverses for preconditioning. Our results will be published in a technical report and, if sufficient positive results are achieved, a journal or conference paper.

2 Background

Linear systems of equations are useful for a wide variety of scientific computing and engineering problems. Common applications include modeling airflow over a wing, heat transfer, traffic flow, electrical networks, and mechanics. Formally, a linear system of equations is a set of n linear equations in k unknowns. For computations linear systems are often represented by the matrix equation $Ax = b$ where A is the matrix of coefficients, x is the vector of unknown variables, and b is the vector of solutions. Because problems in this form occur so often in many different fields it is useful to have fast and accurate techniques for solving them on a computer.

The most common way to solve a linear system numerically is to use iterative techniques. In effect the computer "guesses" at the solution, determines how far off it is, and then makes a new "guess". The more iterations required to reach the answer, the longer it takes to solve the system. Often solving the system will take longer than is desirable for practical use. Systems which take many iterations to solve are also more susceptible to rounding errors. Not only can a solution take a long time to compute but care must be taken to ensure that the solution is sufficiently accurate for its application. The condition number of a matrix is used to approximate how amenable the system is to an iterative solution. If A has a low condition numbers iterative techniques will probably work well, if it has a high condition number they may not.

Besides iteration, another approach is to precondition the system. Is there something that can be done to the system so that it is easier to compute? For example, let M be a preconditioner. The solution to the system $M^{-1}Ax = M^{-1}b$ is the same as the original system. If $M^{-1}A$ has a lower condition number than A it will likely be easier to solve the preconditioned system than the original. Ideally it would be possible to choose $M^{-1} = A^{-1}$, in which case $M^{-1}A$ is the identity matrix which has a condition number of 1. Unfortunately calculating the matrix inverse directly is computational expensive and not a viable approach for this problem.

Support Theory is aimed at answering questions about preconditioners and linear systems. For what kind of systems do good preconditioners exist? How are they best calculated? What are the bounds of the condition number of the preconditioned system?

3 Problem Statement

One approach to solving these problems with Support Theory has used specialized combinatorial structures such as graph embeddings. Recent work by Boman, Guattery, and Hendrickson (to appear in the *SIAM*

Journal on Matrix Analysis and Applications) has shown that similar structures exist between a matrix and its orthogonal projector (a generalized identity matrix). There is a specific mapping from a factor of A to the orthogonal projector that provides a factor of the generalized inverse of A. Approximations of the matrix inverse are likely good preconditioners with low condition numbers. For practical purposes it is also necessary that this approximate inverse be sparse (most of its elements are 0). Most known algorithms involving linear systems and matrices perform poorly otherwise. However, to date sparse approximations have been found only in special cases.

This project would attempt to do the following:

- Find a way to generate an approximation of the special mapping from a factor of A to the orthagonal projector. This gives a factor of an approximate inverse of A.
- Establish if the resulting approximations are both sparse and can be used for good preconditioners.
- These steps would entail trying different factorizations, formulating algorithms for producing approximations, and programing experiments to test their performance (probably with the Matlab numerical computing environment). These investigations may reveal special cases where good sparse approximations are possible.
- Based on these specific results, determine under which general conditions sparse approximations do or do not exist.
- If generalized conditions for sparse approximations or particularly useful special cases exist, are there good ways to calculate them with a computer?

4 Research Environment

Both Professor Guattery and I plan to be on campus over the summer. We would meet most days for a few hours to discuss the results of the previous day and choose new avenues of investigation. After examining the literature of matrix decompositions for promising factorizations we would use a mixture of theoretical analysis (pencil and paper) and empirical experiments (Matlab) to find which are useful. We hope to be able to predict how decompositions will behave and use this to select better ones or to revise our understanding.

Of the eight weeks, the first six would be dedicated primarily to investigation. The final two would focus on formalizing and documenting our results. No specialized computing systems beyond those already available at Bucknell are needed.

5 Anticipated Results

The results will be documented in a technical report. Showing either that sparse approximations are possible or proving they are not would be useful. If the results are sufficiently interesting a conference or journal paper may be possible.