

- 1) If you subtract 4 from 2004 and divide the result by 2, you get a number that is a perfect cube. What is the next largest number after 2004 that has this property?

- 2) Complex numbers first appeared in the mathematical literature in Gerolamo Cardano's *Ars Magna* in the year 1545, where he posed the following problem. Find two numbers whose sum is 10 and whose product is 40. What are these two numbers?

- 3) The ratio of u to v is twice the ratio of w to x , but only half the ratio of y to z . If the ratio of w to y is 6, what is the ratio of x to z ?

- 4) If a circle and a square have the same perimeter, what is the ratio of the area of the circle to the area of the square?

- 5) The stock of internet company TechCom has dropped 80% from its all time high value. By what percent must the stock increase to return to its former high value?

- 6) Find a number from which, if you take its square, there shall remain the greatest difference possible. [This problem appeared on a Cambridge University examination in the year 1785.]

- 7) Find a point on the y -axis that is equidistant from $(5, -5)$ and $(1, 1)$.

8) Compute the sum of these ten terms:

$$\sum_{n=1}^{10} \frac{2n+1}{n^2(n+1)^2} = \frac{3}{4} + \frac{5}{36} + \frac{7}{144} + \dots + \frac{21}{12100}.$$

9) A ball of radius 3 inches rests in a V-shaped trough that has a dihedral angle of 60° . How far is the center of the ball from the bottom of the trough?

10) Rita walks towards Kara's house at the rate of R mph. Starting at the same time, Kara walks toward Rita's house at the rate of K mph. The houses are M miles apart. When Rita and Kara meet, how far from Rita's house are they?

11) Let $i = \sqrt{-1}$. If w is a solution of $x^2 + ix - 1 = 0$, what is w^6 ?

12) If x is the answer to problem 3), and y is the answer to problem 3) on last year's John Gold exam, what is $\frac{(x^3 - y^3)(2x + 2y)}{(3x^2 - 3y^2)(x^2 + xy + y^2)}$?

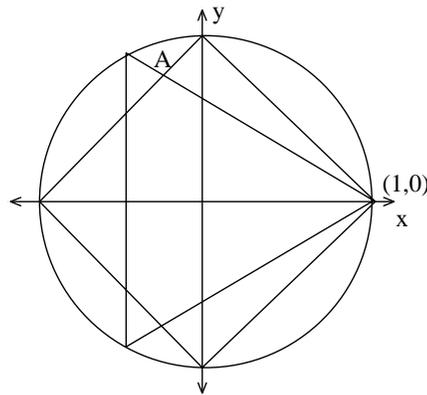
13) If each edge of an equilateral triangle has length 1, what is the distance from the circumcenter of this triangle to any edge? [The circumcenter of a triangle is the point at which the perpendicular bisectors of the three sides meet.]

14) Let $x = \frac{\sin^2 2\theta}{4\cos^2 \theta} + \frac{1}{\sec^2 \theta}$, where $\theta = 2.004$ radians.

Compute $2004x^{10}$.

15) Suppose m and n are distinct natural numbers with the property that their least common multiple is the square of their greatest common divisor. Find the product mn , given that $600 < mn < 900$.

16) A square and an equilateral triangle are inscribed in the circle $x^2 + y^2 = 1$, each with a vertex at $(1, 0)$. What is the x -coordinate of the intersection labeled "A"?



17) How many 5-digit numbers are there whose digits decrease from left to right?

18) What is the value of this infinitely nested expression:

$$\sqrt{3 + \sqrt{3 + \sqrt{3 + \dots}}} ?$$

19) Two equally skilled archers take turns aiming at a target. Each arrow has a 50-50 chance of striking the center. What is the probability that the archer who goes first will be the first to place an arrow in the center?

20) Let $i = \sqrt{-1}$. Compute the infinite sum $1 + \frac{i}{2} + \left(\frac{i}{2}\right)^2 + \left(\frac{i}{2}\right)^3 + \dots$.

Write your answer in the form $a + bi$.

21) A goat is tethered to a corner of a square building by a long leash of length 50 feet. What is the area of the ground that the goat may graze, if the building is 30 x 30 feet?

22) If a fair coin is tossed repeatedly, what is the probability of producing 3 heads before 2 tails?

23) Aaron and Jason drive from point A to point B, departing from A at the same time. Aaron drives due north 50 miles on I-79 then due east 50 miles to point B on I-80 at 60 mph. Jason drives diagonally northeast from A to B on a state highway. How fast must Jason drive on the state highway in order to reach B at the same time Aaron does?

24) “And thus when a man goes walking his head travels farther than his feet.” – Galileo. Galileo was referring to the circular arcs traveled by head and feet as one walks on the surface of the spherical earth. Suppose a man, 6 feet tall, walks 100 miles along a great circle of the earth. By how many feet does the arc traveled by the top of the man’s head exceed 100 miles? Assume the earth is a sphere with radius 4000 miles.

25) Joanne buys a new car whose tank is full of gasoline. She always refills the tank when it is $\frac{3}{4}$ ths empty. After five such refills, what fraction of the original gasoline remains in the tank? Assume the gasoline in the tank remains thoroughly mixed.

26) At the Pizza Hut in Lewisburg there is a clock whose hands revolve at a rate 20 times faster than those of accurate clocks. Assume that the Pizza Hut clock and an ordinary 12-hour clock are started together at midnight. To the nearest minute, how many minutes elapse before the two clocks agree again?

27) A 25 foot ladder rests against the side of a building with the base of the ladder 7 feet from the building. How far will the base of the ladder be from the building if the top of the ladder slides down 4 feet?

28) For each real number x , let $f(x)$ be the minimum of the numbers $3x$, $x + 1$, and $4 - x$. What is the maximum value of $f(x)$?

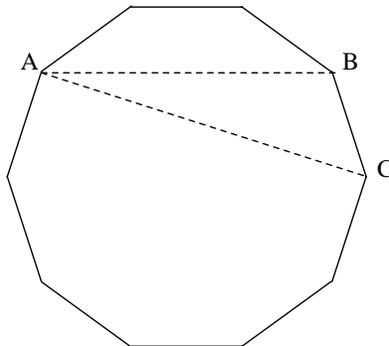
29) Find the area of the quadrilateral with vertices at $(0, 1)$, $(5, -1)$, $(8, 3)$, and $(5, 8)$.

30) A hypersphere is the set of all points in 4-dimensional space equidistant from a fixed point. If the radius of a hypersphere is tripled, by what factor is its hypervolume increased?

31) The center of a circle of radius 7 is 13 units from the center of a circle of radius 2. Line L is tangent to both circles. What is the distance between the two points of tangency?

32) A circular sector of radius r and angle α has twice the area as a sector of radius R and angle β . Find the ratio $\frac{\alpha}{\beta}$ if $R = 2r$.

33) Given this regular 10-sided polygon, what is the measure in degrees of the angle BAC?



34) The length of an interior diagonal of a $4 \times 5 \times 7$ box equals the length of an interior diagonal of a certain cube. What is the length of a side of this cube?

35) Aaron's antique car has a flat windshield with a pair of windshield wipers that are 2 feet long. The pivots of the two wipers are 2 feet apart, and the wipers clean overlapping semicircular spaces. What is the total area of the windshield that is cleaned by wipers?

36) A box contains 2 pennies, 3 nickels, and 4 dimes. Four coins are removed with each coin having an equal probability of being chosen. What is the probability that the value of the four chosen coins is at least 30 cents?

Frank Harary on George Polya:

I must mention that George Polya is not only a distinguished gentleman but a most kind and gentle man: his ebullient enthusiasm, the twinkle in his eye, his tremendous curiosity, his generosity with his time, his spry energetic walk, his warm genuine friendliness, his welcoming visitors into his home and showing them his pictures of great mathematicians he has known – these are all components of his happy personality. As a mathematician, his depth, speed, brilliance, versatility, power, and universality are all inspiring. Would that there were a way of teaching and learning these traits!

George Polya on John von Neumann:

Von Neumann is the only student I was ever intimidated by. He was so quick. There was a seminar for advanced students in Zurich that I was teaching and von Neumann was in the class. I came to a certain theorem, and I said it was not proved and it may be difficult. Von Neumann didn't say anything but after five minutes he raised his hand. When I called on him he went to the blackboard and proceeded to write down the proof. After that I was afraid of von Neumann.

[From the October 1989 *American Mathematical Monthly*]